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1996 J. Phys. A: Math. Gen. 29 5919

(http://iopscience.iop.org/0305-4470/29/18/021)

J. Phys. A: Math. Gen. **29** (1996) 5919–5934. Printed in the UK

Potential symmetries of a porous medium equation

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Received 13 February 1996, in final form 28 May 1996

Abstract. Potential symmetries, which are not local symmetries, are carried out for the porous medium equation $u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x$ where $n \neq 0$, when it can be written in a conserved form. These symmetries are realized as local symmetries of a related auxiliary system, and lead to the construction of corresponding invariant solutions, as well as to the linearization of the equation by non-invertible mappings.

1. Introduction

The quasi-linear equation

$$u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x \tag{1}$$

corresponds to porous media with sources or thermal evolution with sources and convection. This equation exhibits a wide variety of wave phenomena, some of which were studied for $f(x) =$ constant and $g(x) =$ constant by Rosenau and Kamin [58].

The third term on the right-hand side of (1) is of convective nature. In the theory of an unsaturated porous medium, the convective part represents the effect of gravity.

The second term on the right-hand side describes volumetric absorption, which in the case of plasma is caused by radiation to which the plasma is transparent. There is no fundamental reason to assume the spatial-dependent factors in (1) to be constant. Actually, allowing for their spatial dependence enables us to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for stationary factors like the medium's contamination with another material, or in plasma this may express the impact that solid impurities coming from the walls have on the enhancement of the radiation channel.

When $f(x) = 0$ and $g(x) = 0$ equation (1) becomes

$$u_t = (u^n)_{xx}. \tag{2}$$

A complete group classification for the nonlinear heat equation (2) was derived by Ovsiannikov [53–55] by considering the PDE as a system of PDEs, and by Bluman [9, 13]. A classification for Lie–Bäcklund symmetries was obtained by Bluman and Kumei [9].

The main known exact solutions of nonlinear diffusion (2) are summarized by Hill [30]. In [31–33], Hill *et al* have deduced a number of first integrals for stretching similarity solutions of the nonlinear diffusion equation, and of general high-order nonlinear evolution equations, by two different integration procedures.

King [39] obtained approximate solutions to the porous medium equation (2), integral results for the multi-dimensional nonlinear diffusion equation [40], and determined [38] new

0305-4470/96/185919+16$19.50

results by generalizing known instantaneous source and dipole solutions of N-dimensional radially nonlinear diffusion equations. He also applied generalized Bäcklund transformations and obtained a number of equivalence transformations to derive links between a large number of different types of nonlinear diffusion equations [42, 45].

Nonlinear diffusion with absorption arises in many areas of science and engineering. It occurs in the spatial diffusion processes where the physical structure of the medium changes with concentration. The same PDE also arises in the context of nonlinear heat conduction with a source term. For example, materials undergoing heating by microwave radiation exhibit thermal conductivities and body heating which are strongly dependent on temperature.

If we suppose that the diffusivity and absorption term have a power-law dependence on concentration $u(x, t)$ then the basic equation is

$$u_t = (u^n)_{xx} + g(x)u^m \tag{3}$$

where n and m are constants. In this case, for $g(x) =$ constant, exact solutions and first integrals of (3) were obtained by Hill in [33], by the technique of separation of variables and the use of invariant one-parameter group transformations to reduce the governing PDE to various ODEs. For two of the equations thus obtained, first integrals were deduced which subsequently give rise to a number of explicit simple solutions.

Nonlinear diffusion with absorption is characterized by phenomena such as 'blow-up', 'extinction', and 'waiting-time' behaviour. The indices n and m encompass a wide range of this physical behaviour. For example, Kalashnikov [35] has shown that $u(x, t) \equiv 0$ for all x after a finite time provided that $n > 1$ and $0 < m < 1$, a phenomenon referred to as 'extinction'. A well known exact solution of (3) applying for $m = 2 - n$ is due to Kersner [37].

For $m = 1$, Gurtin and MacCamy [28] proposed a transformation that reduces (3) with $g(x) =$ constant and $m = 1$ to (2). However, in general, the background details necessary to obtain solutions of (3) with $m = 1$ via this transformation and (2) are about the same as those required to obtain the solutions directly from (3).

In [24] Galaktionov presented a technique of 'separation of variables' for constructing new exact solutions of the nonlinear heat conduction equations with a source, which are reduced to equations with quadratic nonlinearities. Most of the solutions thus constructed are not invariant under point transformation groups and Lie–Bäcklund groups. The proposed method was first implemented in [6] to construct an exact solution of equation (1) with $f(x) = 0$, $g(x) \equiv C > 0$ and $m = n$. In [25] a method is proposed to obtain exact blow-up solutions for nonlinear heat conduction equations with source. Several references for the classification of Lie and Lie–Bäcklund symmetries for heat equations, in homogeneous and inhomogeneous media, are also listed in [34].

Equation (1) for $g(x) \equiv 0$ adopts the following form:

$$u_t = (u^n)_{xx} + f(x)u^s u_x. \tag{4}$$

For $s = 1$ we obtain a particular case of the generalized Hopf equation. Lie symmetries for this equation were obtained by Katkov [36].

The generalized diffusion equation

$$T_t = (D_1(T)T_x)_x + a(D_2(T))_x + b(x, t)D_3(T,)$$

where $T(x, t)$ denotes the temperature at a point, a is an arbitrary constant, D_1, D_2 and D_3 are arbitrary functions of temperature T and $b(x, t)$ is another arbitrary function of x and t, has been analysed via an isovector approach and some new exact solutions have been obtained by Bhutani [7].

The one-dimensional reaction–diffusion process, governed by a system of nonlinear differential equations with arbitrary source functions

$$a_t = D_1 a_{xx} + A(a, b, x, t)$$
$$b_t = D_2 b_{xx} + B(a, b, x, t)$$

where x and t are space and time coordinates, a and b are the reaction–diffusion variables, $A(a, b, x, t)$ and $B(a, b, x, t)$ are arbitrary nonlinear functions describing the kinetics of the process and $D_1 \neq 0$ and $D_2 \neq 0$ are diffusion constants, is studied by an isovector method; similarity solutions and nonlinear ODEs are provided for fairly general forms of the source functions by Suhubi [61].

Classical and non-classical symmetries of the nonlinear equation (3), with $n = 1$, are considered by Clarkson and Mansfield [19] by using the method of differential Gröbner bases, and by Arrigo *et al* [5] constructing several new exact solutions.

In [26] a group classification problem for equation (1) was solved, by studying those spatial forms which admit the classical symmetry group. Both the symmetry group and the spatial dependence were found through consistent application of the Lie-group formalism. The reduction obtained from the optimal system of subalgebras were derived.

The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of Lie group theory provides the systematic method to search for these special group-invariant solutions. For PDEs with two independent variables, as is equation (1), a single group reduction transforms the PDE into ODEs, which are generally easier to solve than the original PDE. Most of the required theory and description of the method can be found in [11, 30, 51, 55, 61].

Local symmetries admitted by a nonlinear PDE are also useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists. A nonlinear scalar PDE is linearizable by an invertible contact (point) transformation if and only if it admits an infinite-parameter Lie group of contact transformations satisfying specific criteria [10–12, 48].

An obvious limitation of group-theoretic methods based in local symmetries, in their utility for particular PDEs, is that many of these equations does not have local symmetries. It turns out that PDEs can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner. It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations, it is not linearizable by an invertible contact transformation. However, most of the interesting linearizations involve non-invertible transformations; such linearizations can be found by embedding given nonlinear PDEs in auxiliary systems of PDEs [10].

Krasil'shchik and Vinogrod [46, 47, 63] gave criteria which must be satisfied by nonlocal symmetries of a PDE when realized as local symmetries of a system of PDEs which 'covers' the given PDE. Akhatov *et al* [2] gave nontrivial examples of nonlocal symmetries generated by heuristic procedures. By using nonlocal symmetries, some exact solutions which are not similarity solutions of (2) for special values of n were obtained by King [43, 44]. When $n = 1$ in (4) we obtain the Burgers equation. Nonlocal symmetries and Lie–Bäcklund symmetries for this equation are well known [2, 35, 37, 51].

The main purpose of this paper is to obtain potential symmetries for the porous medium equation (1). As far as we know, there are no results on potential symmetries for this equation. In order to find the potential symmetries of (1), we must write this equation in the conserved form. When $m = s + 1$ and $mg(x) = f'(x)$, equation (1) can be written in a conserved form

$$u_t = \left[(u^n)_x + \frac{f(x)}{m}u^m\right]_x. \tag{5}$$

In [10, 11] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by $R\{x, t, u\}$ in a conserved form, a related system denoted by $S\{x, t, u, v\}$ with potentials as additionals dependent variables is obtained. Any Lie group of point transformations admitted by $S\{x, t, u, v\}$ induces a symmetry for $R\{x, t, u\}$; when at least one of the generators of the group depends explicitly on the potential, then the corresponding symmetry is neither a point nor a Lie–Bäcklund symmetry. These symmetries of $R\{x, t, u\}$ are called *potential* symmetries.

The nature of potential symmetries allows one to extend the uses of point symmetries to such nonlocal symmetries. In particular:

(i) Invariant solutions of $S\{x, t, u, v\}$ yield solutions of $R\{x, t, u\}$ which are not invariant solutions for any local symmetry admitted by $R\{x, t, u\}$.

(ii) If $R\{x, t, u\}$ admits a potential symmetry leading to the linearization of $S\{x, t, u, v\}$ then $R\{x, t, u\}$ is linearized by a non-invertible mapping.

In order to find the potential symmetries of (5) we write the equation in a conserved form

$$D_x F - D_t G = 0 \tag{6}$$

where

$$G = u$$

and

$$F = (u^n)_x + \frac{f(x)}{m}u^m.$$

The associated auxiliary system $S\{x, t, u, v\}$ is given by

$$\begin{aligned} v_x &= u \\ v_t &= (u^n)_x + \frac{f(x)}{m}u^m. \end{aligned} \tag{7}$$

Suppose $S\{x, t, u, v\}$ admits a local Lie group of transformations with infinitesimal generator

$$X_S = p(x, t, u, v)\frac{\partial}{\partial x} + q(x, t, u, v)\frac{\partial}{\partial t} + r(x, t, u, v)\frac{\partial}{\partial u} + s(x, t, u, v)\frac{\partial}{\partial v}. \tag{8}$$

This group maps any solution of $S\{x, t, u, v\}$ to another solution of $S\{x, t, u, v\}$ and hence induces a mapping of any solution of $R\{x, t, u\}$ to another solution of $R\{x, t, u\}$. Thus (8) defines a symmetry group of $R\{x, t, u\}$.

If

$$\left(\frac{\partial p}{\partial v}\right)^2 + \left(\frac{\partial q}{\partial v}\right)^2 + \left(\frac{\partial r}{\partial v}\right)^2 \neq 0 \tag{9}$$

then (8) yields a nonlocal symmetry of $R\{x, t, u\}$ such nonlocal symmetry is called a *potential* symmetry of $R\{x, t, u\}$.

Bluman [10] gave theorems which give necessary and sufficient conditions under which nonlinear partial differential equations (scalar or systems) can be transformed to linear PDEs by invertible mappings. In particular such an invertible mapping does not exist if:

(i) a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations;

(ii) a nonlinear system of PDEs does not admit an infinite-parameter Lie group of point transformations.

Suppose $R\{x, t, u\}$ cannot be linearized by an invertible mapping but an associated system $S\{x, t, u, v\}$ admits an infinite-parameter Lie group of point transformations which leads to its linearization by an invertible mapping. Then $R\{x, t, u\}$ is linearized by a non-invertible mapping.

In section 2 we study the classical symmetries of (7) so that all the potential symmetries for (5) are classified. We determine exact solutions of (7) that lead to exact solutions of (5) which cannot be obtained as invariant solutions of its admitted local symmetries. For some special values of n and m it happens that (5) does not admit an infinite-parameter Lie group of contact transformations, so is not linearizable by an invertible mapping; however, the associated system (7) does admit an infinite-parameter Lie group of point transformations and, consequently, equation (5) is linearized by a non-invertible mapping. In these cases, the similarity solutions are discussed in terms of the linearized form. The cases with $f = 0$ represent well known results and we omit them.

In sections 3 and 4 we write, for some functions $f(x)$ $g(x)$ and some parameters n, m and s, respectively (3) and (4) in a conservative form. We study the classical symmetries for the associated system. We obtain some potential symmetries for (3) and (4).

2. Potential symmetries of equation (5)

For $n \neq 0$, if system (7) is invariant under a Lie group of point transformations with infinitesimal generator (8) then

$$\begin{aligned} p &= p(x, t, v) \\ q &= q(t) \\ r &= -p_v u^2 + (s_v - p_x)u + s_x \end{aligned}$$

and

$$s = s(x, t, v).$$

The integrated equation

$$v_t = n(v_x)^{n-1} v_{xx} + \frac{f(x)}{m}(v_x)^m \tag{10}$$

is invariant under a Lie group of point transformations with infinitesimal generator

$$X_E = p(x, t, u, v)\frac{\partial}{\partial x} + q(x, t, u, v)\frac{\partial}{\partial t} + s(x, t, u, v)\frac{\partial}{\partial v}. \tag{11}$$

We can distinguish the following cases depending on n, m and f.

2.1. *Case I: $n \neq 1$, m arbitrary*

Besides the trivial subgroups $X_2 = \frac{\partial}{\partial t}$, $X_4 = \frac{\partial}{\partial v}$, we obtain table 1.

Point symmetries of system (7) project onto local symmetries of equation (5).

2.2. *Case II.a: $n = -1$, m arbitrary*

Besides X_2, and X_4 we obtain table 2. X_1 and X_5 project to point symmetries of (5).

Table 1. Case I: $n \neq 1$.

(a)	p	q	r	s	f	m
X_1	1	0	0	0	c	arbitrary
X_5	x	0	$\frac{2u}{n-1}$	$\frac{n+1}{n-1}v$	$cx^{\frac{n-2m+1}{n-1}}$	arbitrary
X_6	0	t	$\frac{1}{1-n}u$	$\frac{1}{1-n}v$	arbitrary	n

Table 2. $n = -1$, m arbitrary.

(a)	p	q	r	s	f
X_1	1	0	0	0	c
X_5	x	0	$-u$	0	cx^m

Table 3. Case II.b: $n = -1$, $m = 1$.

(a)	p	q	r	s	f
X_5	x	0	$-u$	0	cx
X_6	$-ctx$	t	$ctu + \frac{u}{2}$	$\frac{v}{2}$	cx
X_7	$x\left(\frac{v^2}{4} - \frac{t}{2}\right) - ct^2x$	t^2	$-\frac{uv^2}{4} + \frac{3tu}{2} - \frac{u^2vx}{2} + ct^2u$	tv	cx
X_8	$\frac{xv}{2}$	0	$-\frac{uv}{2} - \frac{xu^2}{2}$	t	cx
X_9	$\frac{be^{bt}xv}{2}$	0	$-\frac{be^{bt}(xu^2 + uv)}{2}$	e^{bt}	$cx - bx\log(x)$
X_{10}	$e^{bt}x$	0	$-e^{bt}u$	0	$cx - bx\log(x)$
X_{11}	$-\frac{1}{c}$	t	$\frac{u}{2}$	$\frac{v}{2}$	be^{cx}
X_∞	$\alpha(v, t)$	0	$-\alpha_v u^2$	0	cx

2.3. *Case II.b:* $n = -1$, $m = 1$

Besides X_2 and X_4 we obtain table 3 where $\alpha(v, t)$ satisfies the following equation:

$$\alpha_{vv} + \alpha_t + c\alpha = 0. \tag{12}$$

X_5, X_6, X_{10} and X_{11} project onto point symmetries of (5), while X_7, X_8, X_9, and X_∞ induce nonlocal (potential) symmetries admitted by (5).

(i) For X_7 and X_8 as the equation readily linearizes, we discuss the similarity solutions in terms of the linearized form.

(ii) For X_9 we obtain the similarity variable

$$z = t \tag{13}$$

Table 4. Case II.b: $n = -1$, $m = -1$, and $f(x) = c$.

(a)	p	q	r	s
X_5	$z_1 z_2$	z_2^2	$\left(cz_2^2 - \frac{z_1^2}{4} - \frac{3z_2}{4}\right)w_2 - \frac{z_1 w_1}{2}$	$\left(cz_2^2 - \frac{z_1^2}{4} - \frac{z_2}{2}\right)w_1$
X_6	$\frac{z1}{2}$	z_2	$(cz_2 - \frac{1}{2})w_2$	$cz_2 w_1$
X_7	z_2	0	$-\frac{1}{2}(z_1 w_2 + w_1)$	$-\frac{z_1 w_1}{2}$
X_8	0	0	w_2	w_1
X_∞	0	0	$\alpha_x(x,t)$	$\alpha(x,t)$

and the similarity solution

$$v = \left(\frac{4\ln(x)}{b} + E(t)\right)^{\frac{1}{2}}. \tag{14}$$

Substituting (14) into (10) we obtain an ODE whose solution is

$$E(z) = k_1 \exp(bt) - \frac{4c}{b^2} - \frac{2}{b} \tag{15}$$

from which we obtain the exact explicit solution

$$v = \left(\frac{4\ln(x)}{b} + k_1 \exp(bt) - \frac{4c}{b^2} - \frac{2}{b}\right)^{\frac{1}{2}}. \tag{16}$$

By means of the first equation of (7)

$$u = 2\left(b^2 x^2 \left(\frac{4\ln(x)}{b} + k_1 \exp(bt) - \frac{4c}{b^2} - \frac{2}{b}\right)\right)^{-\frac{1}{2}}. \tag{17}$$

The nonlinear equation (5) with $n = -1$ and $m = 1$ does not admit an infinite-parameter Lie group of contact transformations; however, its associated auxiliary system (7) admits an infinite-parameter Lie group of point transformations with infinitesimal generator X_∞, where $\alpha(v, t)$ is an arbitrary function satisfying the linear equation (12). One can obtain the invertible mapping

$$\begin{aligned} z_1 &= v \qquad z_2 = t \\ w_1 &= x \qquad w_2 = \frac{1}{u} \end{aligned} \tag{18}$$

which transforms any solution $(w_1(z_1, z_2), w_2(z_1, z_2))$ of the linear system

$$\begin{aligned} \frac{\partial w_1}{\partial z_1} &= w_2 \\ \frac{\partial w_2}{\partial z_1} &= \frac{\partial w_1}{\partial z_2} - cw_1 \end{aligned} \tag{19}$$

to a solution $(u(x,t), v(x,t))$ of the nonlinear system (7) and hence to a solution $u(x,t)$ of (5).

We now study the classical symmetries of (19), and besides the trivial subgroups X_1 and X_2, we obtain table 4 where $\alpha(x,t)$ satisfies the following equation:

$$\alpha_t - \alpha_{xx} - c\alpha = 0.$$

X_6 and X_8 project to point symmetries, X_5 and X_7 induce potential symmetries:

(i) For X_5, solving the characteristic equation leads to the similarity variable

$$z = \frac{z_1}{z_2} \tag{20}$$

and to the similarity solution

$$w_1 = \frac{E(z)}{\sqrt{z_2}} \exp\left(cz_2 - \frac{z^2 z_2}{4}\right) \tag{21}$$

$$w_2 = \frac{2H(z) - zz_2E(z)}{2z_2^{3/2}} \exp\left(cz_2 - \frac{z^2 z_2}{4}\right). \tag{22}$$

Substitution of (21) and (22) into (19) leads to

$$E'(z) - H(z) = 0$$
$$E''(z) = 0.$$

Solving this system we obtain the exact solution

$$w_1 = \frac{k_1 z_1 + k_2 z_2}{\sqrt{z_2^3}} \exp\left(cz_2 - \frac{z_1^2}{4z_2}\right) \tag{23}$$

$$w_2 = \frac{2k_1 z_2 - k_1 z_1^2 - k_2 z_1 z_2}{2z_2^{5/2}} \exp\left(cz_2 - \frac{z_1^2}{4z_2}\right). \tag{24}$$

By (18), we obtain the exact solution in an implicit form of (7):

$$x\sqrt{(t)} - \left(\frac{k_1 v}{t} + k_2\right) \exp\left(ct - \frac{v^2}{4t}\right) = 0 \tag{25}$$

$$u(k_1 v^2 + k_2 t v - 2k_1 t) + 2t^{\frac{5}{2}} \exp\left(\frac{v^2}{4t} - ct\right) = 0. \tag{26}$$

(ii) For X_7, we obtain the similarity variable

$$z = t \tag{27}$$

and the similarity solution

$$w_1 = E(z) \exp\left(-\frac{z_1^2}{4z_2}\right) \tag{28}$$

$$w_2 = \left(H(z) - \frac{z_1 E(z)}{2z_2}\right) \exp\left(-\frac{z_1^2}{4z_2}\right). \tag{29}$$

Substitution of (28) and (29) into (19) leads to

$$H(z) = 0$$
$$2zE'(z) - 2czE(z) + E(z) = 0.$$

Solving this system we obtain the following exact solution of (19):

$$w_1 = \frac{k}{z_2^{1/2}} \exp\left(cz_2 - \frac{z_1^2}{4z_2}\right)$$

$$w_2 = -\frac{kz_1}{2z_2^{3/2}} \exp\left(cz_2 - \frac{z_1^2}{4z_2}\right).$$

By (18), we obtain the following exact solution of (7):

$$v = (2t(-2\ln(x) - \ln(t) + 2ct + 2\ln(k)))^{\frac{1}{2}} \tag{30}$$

$$u = -(2t)^{\frac{1}{2}}(x^2(-2\ln(x) - \ln(t) + 2ct + 2\ln(k)))^{-\frac{1}{2}}. \tag{31}$$

Table 5. Case II.c: $n=-1$, $m=-1$ and $f(x)$ arbitrary.

(a)	p	q	r	s
X_5	he^g	0	$-(hfe^g+1)u$	0
X_6	0	t	$\frac{u}{2}$	$\frac{v}{2}$
X_7	$he^g\left(\frac{v^2}{4}-\frac{t}{2}\right)$	t^2	$-\frac{uv^2}{4}+\frac{3tu}{2}+he^g\left(-\frac{u^2v}{2}+\frac{ftu}{2}-\frac{fuv^2}{4}\right)$	tv
X_8	$\frac{he^g v}{2}$	0	$-he^g\frac{u}{2}(u+fv)-\frac{uv}{2}$	t
X_∞	$\alpha(v,t)e^g$	0	$-e^g(\alpha_v u^2+fu\alpha)$	0

2.4. Case II.c: $n=-1$, $m=-1$ and $f(x)$ arbitrary

Besides the trivial subgroups X_2 and X_4 we obtain table 5 where $\alpha(v,t)$ satisfies the heat equation

$$\alpha_t+\alpha_{vv}=0. \tag{32}$$

$$g(x)=\int f(x)\,\mathrm{d}x$$

and

$$h(x)=\int e^{-g(x)}\,\mathrm{d}x.$$

X_5 and X_6 project onto point symmetries of (5), while X_7, X_8 and X_∞ induce potential symmetries of (5).

For X_7 and X_8 as the equation readily linearizes, we discuss the similarity solutions in terms of the linearized form.

The nonlinear equation (5) with $n=-1$, $m=-1$, does not admit an infinite-parameter Lie group of contact transformations; however, its associated auxiliary system (7) admits an infinite-parameter Lie group of point transformations with infinitesimal generator X_∞, where $\alpha(v,t)$ is an arbitrary function satisfying the linear heat equation (32). One can easily obtain the invertible mapping

$$z_1=v \qquad z_2=t$$
$$w_1=\int e^{-\int f(x)\,\mathrm{d}x}\,\mathrm{d}x \qquad w_2=-\frac{e^{-\int f(x)\,\mathrm{d}x}}{u} \tag{33}$$

which transforms any solution $(w_1(z_1,z_2),w_2(z_1,z_2))$ of the linear system

$$\frac{\partial w_1}{\partial z_1}=w_2$$
$$\frac{\partial w_1}{\partial z_2}=\frac{\partial w_2}{\partial z_1} \tag{34}$$

to a solution $(u(x,t),v(x,t))$ of the nonlinear system (7) and hence to a solution $u(x,t)$ of (5).

The classical symmetries of (34) are the same obtained for (7) for $c=0$, and the infinitesimal generators are listed in table 4. X_6 and X_8 project to point symmetries, X_5 and X_7 induce potential symmetries.

(i) For X_5, solving the characteristic equation leads to the similarity variable

$$z = \frac{z_1}{z_2} \tag{35}$$

and to the the similarity solution

$$w_1 = \frac{E(z)}{\sqrt{z_2}} \exp\left(-\frac{z^2 z_2}{4}\right) \tag{36}$$

$$w_2 = \frac{2H(z) - z z_2 E(z)}{2z_2^{3/2}} \exp\left(-\frac{z^2 z_2}{4}\right). \tag{37}$$

Substitution of (36), (37) into (34) leads to

$$E'(z) - H(z) = 0$$
$$E''(z) = 0.$$

Solving this system we obtain the exact solution

$$w_1 = \frac{k_1 z_1 + k_2 z_2}{\sqrt{z_2^3}} \exp\left(-\frac{z_1^2}{4z_2}\right) \tag{38}$$

$$w_2 = \frac{2k_1 z_2 - k_1 z_1^2 - k_2 z_1 z_2}{2z_2^{5/2}} \exp\left(-\frac{z_1^2}{4z_2}\right). \tag{39}$$

By (33), we obtain the exact solution in an implicit form of (7):

$$h(x)\sqrt{t} - \left(\frac{k_1 v}{t} + k_2\right) \exp\left(-\frac{v^2}{4t}\right) = 0. \tag{40}$$

$$u(k_1 v^2 + k_2 t v - 2k_1 t) + 2t^{\frac{5}{2}} \exp\left(g(x) + \frac{v^2}{4t}\right) = 0. \tag{41}$$

(ii) For X_7, we obtain the similarity variable

$$z = t \tag{42}$$

and the similarity solution

$$w_1 = E(z) \exp\left(-\frac{z_1^2}{4z_2}\right) \tag{43}$$

$$w_2 = \left(H(z) - \frac{z_1 E(z)}{2z_2}\right) \exp\left(-\frac{z_1^2}{4z_2}\right). \tag{44}$$

Substitution of (43) and (44) into (34) leads to

$$H = 0$$
$$2zE'(z) + E(z) = 0.$$

Solving this system we obtain the exact solutions

$$w_1 = \frac{k}{\sqrt{z_2}} \exp\left(-\frac{z_1^2}{4z_2}\right) \tag{45}$$

$$w_2 = -\frac{k z_1}{2z_2^{3/2}} \exp\left(-\frac{z_1^2}{4z_2}\right). \tag{46}$$

By (33), we obtain the exact solution of (7)

$$v = 2(t(-\ln(h(x)) + \ln(k) - \ln(\sqrt{t})))^{\frac{1}{2}} \tag{47}$$

$$u = \frac{t^{\frac{1}{2}} h_x}{h(-\ln(h(x)) + \ln(k) - \ln(\sqrt{t}))^{\frac{1}{2}}}. \tag{48}$$

Table 6. Case II.d: $n = -1$, $m = -2$.

(a)	p	q	r	s	f
X_5	$(x+a)\left(\frac{v}{6c} + \frac{\ln(x+a)}{4}\right)$	t	$-\frac{(x+a)u^2}{6c} - \frac{uv}{6c} + \frac{u}{4}[1 - \ln(x+a)]$	$\frac{v}{2} + \frac{t}{3c}$	$\frac{c}{(x+a)^2}$
X_6	$x+a$	0	$-u$	0	$\frac{c}{(x+a)^2}$

2.5. *Case II.d: $n = -1$, $m = -2$*

Besides X_2, and X_4, we obtain table 6. In this case X_5 induces a nonlocal (potential) symmetry of (5), while X_3 projects onto point symmetries.

For X_5, solving the characteristic equation leads to the similarity variable

$$z = \frac{v}{\sqrt{t}} - \frac{2\sqrt{t}}{3c} \tag{49}$$

and to the normal solution in an implicit form

$$x = \exp\left(\frac{2\sqrt{t}z}{3c} + \frac{4t}{27c^2} + \frac{t^{\frac{1}{4}}}{E(z)}\right) - a. \tag{50}$$

Substitution of equation (50) into the integrated equation (10), leads to the symmetry reduction

$$4E(z)^4E''(z) - 2zE(z)^4E'(z) - 2cE'(z)^3 - 8E(z)^3E'(z)^2 - E(z)^5 = 0. \tag{51}$$

Unfortunately we do not know the solution for this ODE, but the implicit ansatz for the solution is

$$E(z)\left(\frac{v}{\sqrt{t}} - \frac{2\sqrt{t}}{3c}\right)(18cv - 27c^2\ln(x+a) - 8t) + 27c^2t^{\frac{1}{4}} = 0. \tag{52}$$

We must note that the similarity variables which depend on the dependent variable v such as (49) are unusual and cannot be obtained by the direct method of Clarkson and Kruskal [18], as was pointed out by King [42], but could be obtained by the direct method extension due to Arrigo *et al* [4].

2.6. *Case III.a: $n = 1$, m arbitrary*

Besides X_2, and X_4 we obtain table 7. Here X_5 and X_6 project onto point symmetries of (5).

Table 7. Case III.a: $n = 1$, m arbitrary.

(a)	p	q	r	s	f
X_1	1	0	0	0	c
X_5	$\frac{x}{2}$	t	$-\frac{u}{2}$	0	cx^{m-2}
X_6	0	0	u	v	0

2.7. *Case III.b:* $n = 1$, $m = 2$

We have obtained potential symmetries only for f = constant; in that case equation (5) becomes the Burgers equation and we omit the results as they are well known.

2.8. *Case III.c:* $n = 1$, $m = 3$

Besides the symmetries corresponding to $n = 1$, m arbitrary we obtain table 8. X_5 projects onto a point symmetry of (5).

Table 8. Case III.c: $n = 1$, $m = 3$

(a)	p	q	r	s	f
X_5	$\frac{ax+b}{2a}$	t	$-\frac{d+1}{4}u$	$-\frac{d-1}{4}v$	$(ax+b)^d$

2.9. *Case III.d:* $n = 1$, $m = 1$

Equation (5) becomes the Fokker–Planck equation

$$u_t = u_{xx} + (f(x)u)_x = 0.$$

The potential symmetries for the natural potential system

$$\begin{aligned} v_x &= u \\ v_t &= u_x + f(x)u. \end{aligned} \tag{53}$$

were classified by Pucci and Saccomandi [56].

3. Potential symmetries for equation (3)

Equation (3) corresponds to nonlinear diffusion with absorption. Equation (3) for $m = n$ becomes

$$u_t = (u^n)_{xx} + g(x)u^n. \tag{54}$$

In order to find potential symmetries of (54), we write this equation in the conserved form (6), where

$$\begin{aligned} G &= w(x)u \\ F &= w(x)(u^n)_x - w'(x)u^n \end{aligned}$$

and $w(x)$ satisfies

$$w''(x) + g(x)w(x) = 0.$$

3.1. *Case IV.a:* $f = 0$, $n = m$, $n \neq 1, -1$

Besides X_2 and X_4 we obtain table 9. X_1, X_5 and X_6 project onto point symmetries.

3.2. *Case IV.b:* $f = 0$, $n = m$, $g(x) = \int \frac{\mathrm{d}x}{w(x)}$, $w(x)$ *arbitrary,* $n = -\frac{1}{3}$

Besides X_2 and X_4 we obtain table 10. X_1, X_5 and X_6 project onto point symmetries.

Table 9. Case IV.a: $f = 0$, $m = n$, $n \neq 1, -1$.

(a)	p	q	r	s	$w(x)$
X_1	1	0	0	0	c
X_5	x	0	$\dfrac{2u}{n-1}$	0	$cx^{\frac{1+n}{1-n}}$
X_6	0	$(1-n)t$	u	v	arbitrary

Table 10. Case IV.b: $f = 0$, $m = n$, $g(x) = \int \frac{1}{w^2}$, w arbitrary, $n = -\frac{1}{3}$.

(a)	p	q	r	s
X_1	$-\dfrac{2g}{g'}$	0	$\left(3 - \dfrac{3gg''}{g'^2}\right)u$	v
X_5	$\dfrac{3g}{2g'}$	t	$\left(\dfrac{9gg''}{4g'^2} - \dfrac{3}{2}\right)u$	0
X_6	$\dfrac{1}{g'}$	0	$\dfrac{3g''}{2g'^2}u$	0

Table 11. Case IV.c: $f = 0$, $m = n = -1$, $w(x) = \frac{c^2(x+b)^2}{4}$.

(a)	p	q	r	s
X_5	$(x+b)v$	0	$-\dfrac{c^2(x+b)^3u^2}{4} - uv$	v^2
X_6	$\dfrac{x+b}{2}$	0	$-\dfrac{u}{2}$	v
X_7	$-\dfrac{x+b}{4}$	t	$\dfrac{3u}{4}$	0

3.3. Case IV.c: $f = 0$, $n = m = -1$, $w(x) = \frac{c^2(x+b)^2}{4}$

Besides X_2 and X_4 we obtain table 11. We obtain symmetries X_5 and X_6 that project onto point symmetries of (54) and the potential symmetry X_6. $w(x) = \frac{c^2(x+b)^2}{4}$, that is $g(x) = -\frac{2}{(x+b)^2}$. For X_6 we obtain the similarity variable

$$z = t$$

and the similarity solution

$$v = E(t)(x+b).$$

Substituting the similarity solution into (10) we obtain an ODE whose solution is

$$E(z) = k$$

from which we obtain the exact solution

$$v = k(x+b)$$

$$u = \frac{4k}{c^2(x+b)^2}.$$

We must note that this solution, although trivial, cannot be obtained by means of Lie classical symmetries.

4. Potential symmetries for equation (4)

Equation (4) corresponds to nonlinear diffusion with convection. Equation (4) for $s = n - 1$ becomes

$$u_t = (u^n)_{xx} + \frac{f(x)}{n}(u^n)_x. \tag{55}$$

In order to find potential symmetries of (55), we write this equation in the conserved form (6), where

$$G = \frac{u}{f(x)}$$
$$F = \frac{(u^n)_x}{f(x)} + cu^n.$$

4.1. Case V: $g = 0$, $s = n - 1$.

Besides X_2 and X_4 we obtain:

Table 12. Case V: $g = 0$ and $s = n - 1$.

(a)	p	q	r	s	n	$f(x)$
X_1	1	0	0	0	n arbitrary	k
X_5	$\frac{(n-1)(ax+b)}{2an}$	0	$\frac{u}{n}$	v	arbitrary	$\frac{k}{c(ax+b)}$
X_6	$\frac{ax+b}{2an}$	t	$-\frac{u}{n}$	0	n arbitrary	$\frac{k}{c(ax+b)}$
X_7	$(n+1)^{\frac{1-n}{1+n}}(ax+b)^{\frac{1-n}{1+n}}$	0	$-2an+1^{-\frac{2n}{n+1}}(ax+b)^{-\frac{2n}{n+1}}u$	0	$\neq -1$	$\frac{2n}{(n+1)c(ax+b)}$
X_8	$\frac{2}{a}(ax+b)v$	0	$\frac{4c(ax+b)^2u^2}{3a^2} - 2uv$	v^2	-1	$-\frac{3}{2c(ax+b)}$
X_9	0	$(1-n)t$	u	v	arbitrary	arbitrary

Here X_5, X_6, X_7 and X_9 project onto point symmetries of (55); while X_8 induces potential symmetries admitted by (55).

Solving the characteristic equation we obtain the similarity variable $z = t$ and similarity solution $v = \sqrt{ax+b}E(t)$ where $E(t)$ satisfies $E'(t) = 0$ so $E =$ constant. We obtain the trivial solution

$$v = C_1\sqrt{ax+b}$$
$$u = -\frac{3a^2C_1}{4c(ax+b)^{\frac{3}{2}}}.$$

We must note that although the infinitesimal p depends explicity on v the similarity variable does not depend on v; however the solution obtained is not invariant under a Lie group of point transformations.

5. Concluding remarks

In this paper we have classified the potential symmetries of the quasi-linear parabolic equation (5). Potential symmetries of (5) are found by studying the classical point symmetries of the auxiliary system (7). Recognizing the importance of the space-dependent parts on the overall dynamics of (5), we have studied the different choices for the function $f(x)$, and constants n and m, for which system (7) is invariant under a Lie group of point transformations, as well as their infinitesimal generators. Consequently we have obtained the class of functions $f(x)$ as well as the constants n and m for which equation (5) admits potential symmetries. By using these symmetries we have found similarity solutions of the auxiliary system (7) that yield to exact solutions of (5) which cannot be obtained by classical Lie symmetries. Some of these solutions appear in implicit form, and as the similarity variable depends on the dependent variable, cannot be obtained by the direct method of Clarkson and Kruskal either.

The nonlinear equation (1), with $n = -1$, $m = 1$ and $n = -1$, $m = -1$, does not admit an infinite-parameter Lie group of contact transformations, so cannot be linearizable by an invertible contact (point) transformation. We have constructed nonlocal symmetries (potential symmetries) which are realized as local symmetries of a related auxiliary system of differential equations, by using potential symmetries we have also linearized (1) by an explicit non-invertible mapping.

In a forthcoming work, a method based on the 'nonclassical symmetries' due to Bluman and Cole [12], will be used to obtain new solutions to (3), the new solutions being unobtainable by the method of Lie classical symmetries or potential symmetries.

Acknowledgments

It is a pleasure to thank Professor Philip Rosenau for bringing the porous medium equation to my attention, as well as for his illuminating discussions and great help and support on this work. I am also grateful to Professors G W Bluman, E Pucci, P Clarkson, V A Dorodnitsyn and J R King for sending me their preprints, as well as Professor J L Romero for his useful suggestions. I also want to thank the referees for their helpful comments.

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